Friday, September 18, 2015

Page 390

Problem 1

Problem. Evaluate the functions.

(a) $\sinh 3$

(b) $\tanh(-2)$

Solution. (a)

$$\begin{aligned}\sinh 3 &= \frac{e^3 - e^{-3}}{2} \\ &= 10.0178\ldots.\end{aligned}$$

Or use the `sinh` function on the TI-83 and get the same answer.

(b)

$$\begin{aligned}\tanh -2 &= \frac{e^{-2} - e^{2}}{2} \\ &= -0.96402\ldots.\end{aligned}$$

Or use the `tanh` function on the TI-83 and get the same answer.

Problem 5

Problem. Evaluate the functions.

(a) $\cosh^{-1} 2$

(b) $\operatorname{sech}^{-1} \frac{2}{3}$

Solution. (a)

$$\begin{aligned}\cosh^{-1} 2 &= \ln(2 + \sqrt{2^2 - 1}) \\ &= \ln 3.73205 \\ &= 1.3169.\end{aligned}$$

Or use the `cosh`$^{-1}$ function the TI-83 and get the same answer.

(b) Use the fact that the "angle" whose hyperbolic secant is $\frac{2}{3}$ is the same as the "angle" whose hyperbolic cosine is $\frac{3}{2}$. Then

$$\begin{aligned}
\operatorname{sech}^{-1}\frac{2}{3} &= \cosh^{-1}\frac{3}{2} \\
&= \ln\left(\frac{2}{3} + \sqrt{\left(\frac{2}{3}\right)^2 - 1}\right) \\
&= \ln 2.618033. \qquad\qquad = 0.96242.
\end{aligned}$$

There is no `sech` or `sech`$^{-1}$ function on the TI-83.

Problem 7

Problem. Verify the identity $\tanh^2 x + \operatorname{sech}^2 x = 1$.

Solution. Use the definitions and simplify.

$$\begin{aligned}
\tanh^2 x + \operatorname{sech}^2 x &= \left(\frac{e^x - e^{-x}}{e^x + e^{-x}}\right)^2 + \left(\frac{4}{e^x + e^{-x}}\right)^2 \\
&= \frac{e^{2x} - 2 + e^{-2x} + 4}{(e^x + e^{-x})^2} \\
&= \frac{e^{2x} + 2 + e^{-2x}}{e^{2x} + 2 + e^{-2x}} \\
&= 1.
\end{aligned}$$

This is reminiscent of the identity

$$\tan^2 x - \sec^2 x = 1.$$

Problem 11

Problem. Verify the identity $\sinh 2x = 2\sinh x \cosh x$.

Solution. Simplify the right-hand side and see that it matches the definition of $\sinh 2x$.

$$\begin{aligned}
2\sinh x \cosh x &= 2 \cdot \left(\frac{e^x - e^{-x}}{2}\right) \cdot \left(\frac{e^x + e^{-x}}{2}\right) \\
&= \frac{(e^x - e^{-x})(e^x + e^{-x})}{2} \\
&= \frac{(e^x)^2 - (e^{-x})^2}{2} \\
&= \frac{e^{2x} - e^{-2x}}{2} \\
&= \sinh 2x.
\end{aligned}$$

This is reminiscent of the identity

$$\sin 2x = 2 \sin x \cos x.$$

Problem 13

Problem. Verify the identity $\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$.

Solution. Begin with the right side and simplify it.

$$\begin{aligned}
\sinh x \cosh y + \cosh x \sinh y &= \left(\frac{e^x - e^{-x}}{2}\right)\left(\frac{e^y + e^{-y}}{2}\right) + \left(\frac{e^x + e^{-x}}{2}\right)\left(\frac{e^y - e^{-y}}{2}\right) \\
&= \frac{(e^x - e^{-x})(e^y + e^{-y})}{4} + \frac{(e^x + e^{-x})(e^y - e^{-y})}{4} \\
&= \frac{(e^{x+y} + e^{x-y} - e^{-x+y} - e^{-x-y})}{4} + \frac{(e^{x+y} - e^{x-y} + e^{-x+y} - e^{-x-y})}{4} \\
&= \frac{(2e^{x+y} - 2e^{-x-y})}{4} \\
&= \frac{(e^{x+y} - e^{-x-y})}{} \\
&= \sinh(x+y)
\end{aligned}$$

This is reminiscent of the identity

$$\sin(x+y) = \sin x \cos y + \cos x \sin y.$$

Problem 17

Problem. Find the limit $\lim_{x\to\infty} \sinh x$.

Solution.

$$\begin{aligned}\lim_{x\to\infty}\sinh x &= \lim_{x\to\infty}\frac{e^x - e^{-x}}{2}\\ &= \lim_{x\to\infty}\frac{e^x}{2} - \lim_{x\to\infty}\frac{e^{-x}}{2}\\ &= \infty - 0\\ &= \infty.\end{aligned}$$

Problem 23

Problem. Find the derivative of $f(x) = \sinh 3x$.

Solution.

$$\begin{aligned}f'(x) &= \cosh 3x \cdot 3\\ &= 3\cosh 3x.\end{aligned}$$

Problem 24

Problem. Find the derivative of $f(x) = \cosh(8x+1)$.

Solution.

$$\begin{aligned}f'(x) &= \sinh(8x+1)\cdot 3\\ &= 3\sinh(8x+1).\end{aligned}$$

Problem 27

Problem. Find the derivative of $f(x) = \ln(\sinh x)$.

Solution.

$$\begin{aligned}f'(x) &= \frac{\frac{d}{dx}(\sinh x)}{\sinh x}\\ &= \frac{\cosh x}{\sinh x}\\ &= \coth x.\end{aligned}$$

Problem 31

Problem. Find the derivative of $f(t) = \arctan(\sinh t)$.

Solution.

$$\begin{aligned} f'(t) &= \frac{\frac{d}{dx}(\sinh t)}{1+\sinh^2 t} \\ &= \frac{\cosh t}{1+\sinh^2 t} \\ &= \frac{\cosh t}{\cosh^2 t} \\ &= \frac{1}{\cosh t} \\ &= \operatorname{sech} t. \end{aligned}$$

Problem 32

Problem. Find the derivative of $g(x) = \operatorname{sech}^2 3x$.

Solution.

$$\begin{aligned} g'(x) &= 2\operatorname{sech} 3x \cdot \frac{d}{dx}(\operatorname{sech} 3x) \\ &= 2\operatorname{sech} 3x \cdot (-\operatorname{sech} 3x \tanh 3x \cdot 3) \\ &= -6\operatorname{sech}^2 3x \tanh 3x. \end{aligned}$$

Problem 37

Problem. Find any relative extrema of the function $f(x) = \sin x \sinh x - \cos x \cosh x$.

Solution. Find $f'(x)$.

$$\begin{aligned} f'(x) &= (\cos x \sinh x + \sin x \cosh x) - ((-\sin x)\cosh x + \cos x \sinh x) \\ &= \cos x \sinh x + \sin x \cosh x + \sin x \cosh x - \cos x \sinh x \\ &= 2\sin x \cosh x. \end{aligned}$$

Now solve $f'(x) = 0$. Note that because $\cosh x$ never equals 0, we may divide by it.

$$\begin{aligned} 2\sin x \cosh x &= 0, \\ \sin x &= 0, \\ x &= k\pi. \end{aligned}$$

for $k = 0, \pm 1, \pm 2, \ldots$.

Let's use the Second Derivative Test.

$$f''(x) = \cos x \cosh x + \sin x \sinh x.$$

Then, at the critical points $k\pi$, we have

$$\begin{aligned} f''(k\pi) &= \cos k\pi \cosh k\pi + \sin k\pi \sinh k\pi \\ &= \cos k\pi \cosh k\pi \\ &= \pm \cosh k\pi. \end{aligned}$$

We see that $f''(k\pi) > 0$ when k is even (because $\cos k\pi = 1$ and cosh is always positive) and $f''(k\pi) < 0$ when k is odd (because $\cos k\pi = -1$). Thus, $f(x)$ has a relative minimum at $x = 2k\pi$ and a relative maximum at $x = (2k+1)\pi$ for $k = 0, 1, 2, \ldots$.